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Fixed Points for Multivalued Mappings Defined on Unbounded Sets in Banach Spaces

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We give some fixed points theorems for multivalued nonexpansive and pseudocontractive mappings defined on closed convex unbounded subsets of a Banach space. Some of such results extend to the multivoque case a few of the theorems already known for the univoque case, while some other results improve a few of the theorems already known for the multivoque case.

In this paper we study the existence of fixed points for multivalued mappings $T: K \to X$, where K is an unbounded closed convex subset of a real Banach space X and T is either a nonexpansive or a pseudocontractive mapping (see below for their definitions).

Our investigation is prompted by a work of Kirk–Ray [1] in which similar problems are treated for single-valued mappings in uniformly convex Banach spaces and by two recent papers of Carbone–Marino [2] and Marino–Pietramala [3] in which (always for single-valued case) the structure of some geometric sets in Banach spaces is examined.

The theorem of Lim [4] always assures the existence of a fixed point for nonexpansive mappings $T: K \to \mathscr{K}(K)$ from the closed bounded convex subset K of an uniformly convex Banach space X into the family of non-empty compact subsets of K, while it is known that for a wide class of unbounded closed convex sets K (e.g., subsets of Hilbert spaces which contain an infinite ray) there exist nonexpansive mappings $T: K \to \mathscr{K}(K)$ which fail to have a fixed point [1].

Thus it is interesting to investigate the existence of fixed points for non-expansive mappings defined on closed convex unbounded subsets in the "rich" structure of Hilbert spaces too.

A line of research in this direction (for the single-valued case) has been initiated by Goebel–Kuczumov [5] and is expanded in [1, 6, 7, 2, 3].

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In these papers it is shown that it is sufficient to assume that T moves some points in a bounded "direction" for the existence of fixed points.

More precisely in [5] it is proved that if K is a closed convex subset of l_2 and $T: K \to K$ is nonexpansive (single-valued) for which there exists a point $x \in K$ such that the set

$$LS(x, Tx; K) := \{z \in K : \langle z - x, Tx - x \rangle \geqslant 0\}$$

is bounded, then T has a fixed point in K.

On the same line in [1] it is shown that if K is an unbounded closed convex subset of an uniformly convex space X and $T: K \to K$ is lipschitzian pseudocontractive (single-valued) mapping for which the set

$$G(x, Tx; K) := \{z \in K : \|z - Tx\| \leqslant \|z - x\|\}$$

is bounded for some $x \in K$, then T has a fixed point in K.

The present paper seems to be the first to extend to Banach spaces and multivalued mappings the ideas contained (for single-valued case) in [5, 1].

1. Notation and Definitions

In this section we introduce some necessary notation and definitions.

Let X be a real Banach space and let K be a nonempty convex subset of X. We denote by $\mathscr{K}(X)$ (resp. $\mathscr{K}(K)$) the family of nonempty compact subsets of X (resp. K) and by $\mathscr{CB}(X)$ the family of nonempty bounded closed subsets of X.

For any $A \in \mathscr{CB}(X)$ we note with $\overline{\text{co}}(A)$ the closed convex hull of A and for any $A, B \in \mathscr{CB}(X)$ the *Hausdorff metric* H, induced by the norm of X, is defined as

$$H(A, B) := \max(\sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B)),$$

where $d(x, Y) := \inf\{\|x - y\| : y \in Y\}$.

A multivalued mapping $T: K \to \mathscr{CB}(X)$ is said to be lipschitzian if

$$H(Tx, Ty) \leqslant L \|x - y\|$$

for any $x, y \in K$, $L \geqslant 0$. T is *nonexpansive* if $L = 1$ and a *contraction* if $L < 1$.

T is said to be *demiclosed* (on the convex subset K) if x_n converges weakly to x, y_n converges to y, $y_n \in Tx_n$, imply $y \in Tx$.

T is *semiconvex* (on K) if for any $x, y \in K$, $z = \lambda x + (1 - \lambda)y$ $(0 \leqslant \lambda \leqslant 1)$, and any $u \in Tx$, $v \in Ty$ there exists $\eta \in Tz$ such that $\|\eta\| \leqslant \max\{\|u\|, \|v\|\}$.

We introduce now the geometric sets which play a fundamental role in our results. We set (for any $x, y \in X$)

$$\tau(x, y) := \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}.$$

The following properties of τ are well known [8]

$$\tau(ax, y) = \tau(x, y) \qquad (a > 0) \tag{1}$$

$$\tau(x, y) \leqslant \|y\| \tag{2}$$

$$\tau(x, bx + cy) = b\,\|x\| + c\tau(x, y) \qquad (b \text{ real}, c \geqslant 0) \tag{3}$$

$$\tau(x, y + z) \leqslant \tau(x, y) + \tau(x, z) \tag{4}$$

$$\tau(x, y) = \langle x, y \rangle\, \|x\|^{-1} \qquad (\text{in Hilbert spaces}).$$

Following [2] we define, for $x, y \in X$, $\varepsilon > 0$,

$$LS(x, y; K) := \{z \in K : \tau(x - z, y - x) < 0\}$$
$$LS(x, y, \varepsilon; K) := \{z \in K : \tau(x - z, y - x) < \varepsilon\}.$$

In [2] it is shown that, put

$$G(x, y; K) := \{z \in K : \|z - y\| \leqslant \|z - x\|\},$$

X is strictly convex if and only if $G(x, y; X) \subseteq LS(x, y; X)$ for any $x, y \in X$. This remains true if y is replaced by a subset A of X, i.e., X is strictly convex if and only if $G(x, A; X) \subseteq LS(x, A; X)$ for any $x \in X$ and $A \subseteq X$, where G and LS are so defined:

$$LS(x, A; K) := \{z \in K : \exists a \in A : \tau(x - z, a - x) < 0\} = \bigcup_{a \in A} LS(x, a; K)$$

$$LS(x, A, \varepsilon; K) := \{z \in K : \exists a \in A : \tau(x - z, a - x) < \varepsilon\} = \bigcup_{a \in A} LS(x, a, \varepsilon; K)$$

$$G(x, A; K) := \{z \in K : \exists a \in A : \|z - a\| \leqslant \|z - x\|] = \bigcup_{a \in A} G(x, a; K).$$

2. Nonexpansive Mappings

Out first result extends [2, Theorem 1.3] concerning self-mappings to multivalued mappings.

THEOREM 1. *Let X be a real Banach space whose bounded closed convex subsets have the fixed point property for multivalued nonexpansive point-compact self-mappings.*

Let K be a closed convex subset of X and $T: K \to \mathscr{K}(K)$ nonexpansive. If there exists $x_0 \in K$ such that $LS(x_0, \overline{\text{co}}(Tx_0); K)$ is bounded, then T has a fixed point in K.

Prof. Assume $x_0 \notin Tx_0$ (otherwise we are done). It is enough to show that a bounded closed convex subset of K exists which is invariant under T.

We set

$$R := 4 \sup\{H(\{z\}, \overline{\text{co}}(Tx_0)) : z \in LS(x_0, \overline{\text{co}}(Tx_0); K)\}$$

$$S := \{z \in K : \exists v \in \overline{\text{co}}(Tx_0) \text{ such that } \|z - v\| \leqslant R\}.$$

Then one can verify that S is a nonempty bounded closed convex subset of K. It remains to show that $T(S) \subseteq S$. We consider separately the two cases

$$z \in S \cap LS(x_0, \overline{\text{co}}(Tx_0); K) \qquad \text{and} \qquad z \in S,\ z \notin LS(x_0, \overline{\text{co}}(Tx_0); K).$$

Assume first that $z \in S \cap LS(x_0, \overline{\text{co}}(Tx_0); K)$ and let $\eta \in \overline{\text{co}}(Tx_0)$, $\eta \neq x_0$. Then

$$\tau\left(x_0 - \frac{x_0 + \eta}{2}, \eta - x_0\right) = \tau\left(\frac{x_0 - \eta}{2}, \eta - x_0\right) = (\text{from } (1))$$

$$= \tau(x_0 - \eta, \eta - x_0) = (\text{from } (3)) = -\|x_0 - \eta\| < 0,$$

i.e., $(x_0 + \eta)/2 \in LS(x_0, \overline{\text{co}}(Tx_0); K)$, and by definition of R

$$\frac{R}{4} \geqslant H\left(\left\{\frac{x_0 + \eta}{2}\right\}, \overline{\text{co}}(Tx_0)\right) \geqslant \left\|\frac{x_0 + \eta}{2} - \eta\right\| = \frac{1}{2}\|x_0 - \eta\|,$$

i.e., $\|x_0 - \eta\| \leqslant R/2$. Moreover, $\|z - \eta\| \leqslant R/4$ since $z \in LS(x_0, \overline{\text{co}}(Tx_0); K)$, so

$$\|z - x_0\| \leqslant \|z - \eta\| + \|\eta - x_0\| \leqslant \frac{R}{4} + \frac{R}{2} < R.$$

Now [9, Lemma 2] assures that $H(\overline{\text{co}}(Tz), \overline{\text{co}}(Tx_0)) \leqslant H(Tz, Tx_0)$ and so, from nonexpansivity of T, we have

$$H(\overline{\text{co}}(Tz), \overline{\text{co}}(Tx_0)) \leqslant \|z - x_0\| < R.$$

By definition of Hausdorff metric H it follows that for any $u \in Tz$ there exists $v \in \overline{\text{co}}(Tx_0)$ such that

$$\|u - v\| \leqslant H(\overline{\text{co}}(Tz), \overline{\text{co}}(Tx_0)) < R,$$

i.e., $Tz \subseteq S$.

Assume now that $z \in S$ but $z \notin LS(x_0, \overline{\text{co}}(Tx_0); K)$. Then for any $\eta \in \overline{\text{co}}(Tx_0)$ we have

$$\tau(x_0 - z, \eta - x_0) \geqslant 0$$

and, since $z \in S$, there exists $v \in \overline{\text{co}}(Tx_0)$ such that $\|z - v\| \leqslant R$. Moreover,

$$\begin{aligned}
\|z - x_0\| &= \tau(x_0 - z, x_0 - z) \qquad \text{(from (3))} \\
&= \tau(x_0 - z, v - z) - \tau(x_0 - z, v - x_0) \qquad \text{(from (3))} \\
&\leqslant \tau(x_0 - z, v - z) \\
&\leqslant \|v - z\| \qquad \text{(from (2))} \\
&\leqslant R.
\end{aligned}$$

Finally from

$$H(\overline{\text{co}}(Tz), \overline{\text{co}}(Tx_0)) \leqslant H(Tz, Tx_0) \leqslant \|z - x_0\| \leqslant R$$

follows, as before, that for any $u \in Tz$ there exists $v \in \overline{\text{co}}(Tx_0)$ such that

$$\|u - v\| \leqslant H(\overline{\text{co}}(Tz), \overline{\text{co}}(Tx_0)) \leqslant R,$$

i.e., $Tz \subseteq S$, concluding the proof. ∎

The next theorem extends [3, Theorem 3] to multivalued case.

THEOREM 2. *Let K be a closed convex subset of a real Banach space X, and let $T: K \to \mathscr{K}(X)$ be a nonexpansive mapping which satisfies the following "inwardness" condition: $T(x) \subseteq \overline{I_K(x)}$ for any $x \in K$, where $I_K(x) := \{x + c(u - x) : u \in K, c \geqslant 1\}$ is the "inward set" of x relative to K. Suppose for some bounded set $W \subseteq K$ that the set*

$$LS(W, TW; K) := \bigcap_{w \in W} LS(w, Tw, K)$$

is bounded. Then there exists a bounded sequence $\{x_n\} \subseteq K$ such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Proof. Let $y \in K$ fixed, and for any $\alpha \in (0, 1)$ define $T_\alpha : K \to \mathscr{K}(X)$ by

$$T_\alpha(x) := (1 - \alpha) y + \alpha Tx.$$

It is easy to verify that T_α is a contraction that satisfies the "inwardness condition" $T_\alpha(x) \subseteq \overline{I_K(x)}$ for any $x \in K$ (indeed, by convexity of K convexity of $I_K(x)$ follows for any $x \in K$).

Thus from [10, Theorem 3.4] T_α has a fixed point $x_\alpha \in K$.

Suppose, by contradiction, that the set $\{x_\alpha : \alpha \in (0, 1)\}$ is unbounded. Then it is possible to choose $k \in (0, 1)$ such that

$$\sup_{w \in W} H(\{y\}, Tw) < d(x_k, W) \tag{5}$$

$$\sup_{z \in LS(W,\, TW;\, K)} \|z\| < \|x_k\|. \tag{6}$$

We will prove that $x_k \in LS(W, TW; K)$, i.e., that for any $w \in W$ there exists $\eta_w \in Tw$ such that $\tau(w - x_k, \eta_w - w) < 0$. Indeed, by $x_k \in (1-k)\, y + kTx_k$, it follows that $x_k = (1-k)\, y + k\xi$, $\xi \in Tx_k$.

From nonexpansivity of T there exists $\eta_w \in T(w)$ such that

$$\|\xi - \eta_w\| \leqslant \|x_k - w\| \tag{7}$$

and so

$$\begin{aligned} \tau(w - x_k, \eta_w - w) &= \tau(w - x_k, \eta_w - w + x_k - x_k) \\ &= -\|w - x_k\| + \tau(w - x_k, \eta_w - x_k) \qquad \text{(from (3))} \\ &\leqslant -\|w - x_k\| + \|\eta_w - x_k\| \qquad \text{(from (2))} \\ &= -\|w - x_k\| + \|\eta_w - (1-k)\, y - k\xi \pm k\eta_w\| \\ &\leqslant -\|w - x_k\| + k\, \|\xi - \eta_w\| + (1-k)\, \|\eta_w - y\| \\ &\leqslant (k-1)\, \|x_k - w\| + (1-k)\, \|\eta_w - y\| \qquad \text{(from (7))} \\ &\leqslant (k-1)\, \|x_k - w\| + (1-k)\, H(\{y\}, Tw) \\ &< (k-1)\, \|x_k - w\| + (1-k)\, d(x_k, W) \qquad \text{(from (5))} \\ &\leqslant (k-1)\, \|x_k - w\| + (1-k)\, x_k - w\| = 0, \end{aligned}$$

i.e., $x_k \in LS(W, TW; K)$ contradicting (6).

Thus $M := \sup\{\|x_\alpha - y\|, \alpha \in (0, 1)\} < \infty$ and moreover

$$\begin{aligned} d(x_\alpha, Tx_\alpha) = \inf_{z \in T_{x_\alpha}} \|x_\alpha - z\| &\leqslant \left\| x_\alpha - \frac{x_\alpha - (1-\alpha)\, y}{\alpha} \right\| \\ &= \frac{1-\alpha}{\alpha}\, \|x_\alpha - y\| \leqslant \frac{1-\alpha}{\alpha}\, M \end{aligned}$$

and this last converges to 0 as $\alpha \to 1$. ∎

The following corollary improves [11, Corollary 2].

COROLLARY 3. *Let X, K, T be as in Theorem 2. Suppose that, for some bounded set $W \subseteq K$ and $\varepsilon > 0$, the set*

$$LS(W, TW, \varepsilon; K) := \bigcap_{w \in W} LS(w, Tw, \varepsilon; K)$$

is relatively compact. Then T has a fixed point.

Proof. Let $y \in K$ fixed and $A := \sup\{\|y - \xi\|, \xi \in TW\}$. Then for any α belonging to

$$I := \left(\frac{A-\varepsilon}{A}, 1\right) \cap (0, 1)$$

the mapping defined by $T_\alpha(x) := (1-\alpha)\,y + \alpha Tx$ is a contraction such that $T_\alpha(x) \subseteq \overline{I_K(x)}$ and so, as in Theorem 2, T_α has a fixed point $x_\alpha \in T_\alpha(x_\alpha)$.

We show that $x_\alpha \in LS(W, TW, \varepsilon; K)$ for any $\alpha \in I$.

Indeed, since $x_\alpha \in T_\alpha(x_\alpha)$, for any $w \in W$ there exists $\eta_w \in Tw$ such that

$$\|x_a - \eta_w\| \leqslant H(T_\alpha(x_\alpha), Tw). \tag{8}$$

Hence,

$$\begin{aligned}
\tau(w - x_\alpha, \eta_w - w) &= \tau(w - x_\alpha, \eta_w - x_\alpha + x_\alpha - w)\\
&= -\|x_\alpha - w\| + \tau(w - x_\alpha, \eta_w - x_\alpha) \qquad \text{(from (3))}\\
&\leqslant -\|x_\alpha - w\| + \|\eta_w - x_\alpha\| \qquad \text{(from (2))}\\
&\leqslant -\|x_\alpha - w\| + H(T_\alpha(x_\alpha), Tw) \qquad \text{(from (8))}\\
&\leqslant -\|x_\alpha - w\| + H(T_\alpha(x_\alpha), T_\alpha(w)) + H(T_\alpha(w), Tw)\\
&\leqslant -\|x_\alpha - w\| + \alpha\,\|x_\alpha - w\| + H(T_\alpha(w), Tw)\\
&\qquad \text{(from contractivity of } T_\alpha)\\
&= (\alpha - 1)\,\|x_\alpha - w\| + H(T_\alpha(w), Tw)\\
&\leqslant H(T_\alpha(w), Tw).
\end{aligned}$$

Now, it is a routine calculation to verify that $H(T_\alpha(w), Tw) \leqslant (1-\alpha)A$, and so, for the choice of α, we obtain $\tau(w - x_\alpha, \eta_w - w) < \varepsilon$, i.e., $x_\alpha \in LS(W, TW, \varepsilon; K)$.

By Theorem 2 we have $d(x_\alpha, Tx_\alpha) \to 0$ for $\alpha \to 1$.

Let now $\{x_{\alpha(n)}\}$ be a sequence of $\{x_\alpha\}$ such that $x_{\alpha(n)} \to x_0$ (such a sequence necessarily exists from relative compactness of $LS(W, TW, \varepsilon; K)$). Then

$$\frac{x_{\alpha(n)}}{\alpha(n)} \to x_0, \qquad \frac{x_{\alpha(n)} - (1-\alpha(n))\,y}{\alpha(n)} \in Tx_{\alpha(n)}, \qquad \frac{x_{\alpha(n)} - (1-\alpha(n))\,y}{\alpha(n)} \to x_0$$

and from upper semicontinuity of T follows $x_0 \in Tx_0$. ∎

COROLLARY 4. *Let X be a reflexive real Banach space, $K \subseteq X$ closed convex, $T\colon K \to \mathscr{K}(X)$ a nonexpansive mapping such that $T(x) \subseteq \overline{I_K(x)}$ for any $x \in K$, $I - T$ demiclosed on K.*

Suppose for some bounded set $W \subseteq K$ that the set $LS(W, TW; K)$ is bounded. Then T has a fixed point.

Proof. As in the proof of Theorem 2 there exists a bounded sequence $\{x_n\}$ such that

$$x_n = (1 - \alpha_n)\, y + \alpha_n\, y_n$$

with $y_n \in Tx_n$, y_n bounded also, and $\|x_n - y_n\| = (1 - \alpha_n)\, \|y_n\| \to 0$ as $n \to \infty$.

From reflexivity of X there exists a subsequence $x_{n(j)}$ of $\{x_n\}$ such that $x_{n(j)}$ converges weakly to z, for some $z \in K$. So, since $x_n - y_n \in (I - T)(x_n)$ and $x_n - y_n \to 0$, the demiclosure of $I - T$ assures that $0 \in (I - T)(z)$, i.e., $z \in Tz$. ▮

The next corollary improves [11, Theorem 1].

COROLLARY 5. *Let X, K, T, W be as in Theorem 2 and suppose that there exists $\varepsilon > 0$ such that $LS(W, TW, \varepsilon; K)$ is relatively weakly compact. If $I - T$ is demiclosed or semiconvex on K, then T has a fixed point.*

Proof. *First Case.* $I - T$ demiclosed.

As the proof of Corollary 4, since in this case $\{x_n\} \subseteq LS(W, TW, \varepsilon; K)$ (see Corollary 3).

Second Case. $I - T$ semiconvex.

We know (Corollary 3) that there exists a sequence $\{x_n\} \subseteq LS(W, TW, \varepsilon; K)$ such that $d(x_n, Tx_n) \to 0$.

From [11, Propositions 1 and 2] it follows that for any $r > 0$ the sets

$$H_r := \{x \in K : d(x, Tx) \leqslant r\}$$

are (nonempty) closed convex. Hence H_r are weakly closed for $r > 0$. Moreover the sets

$$\tilde{H}_r := H_r \bigcap \overline{LS(W, TW, \varepsilon; K)}^{\text{weak}}$$

are nonempty and weakly closed in $\overline{LS(W, TW, \varepsilon; K)}^{\text{weak}}$ and obviously have the finite intersection property.

Therefore, by the weak compactness of $\overline{LS(W, TW, \varepsilon; K)}^{\text{weak}}$ we have $\bigcap \tilde{H}_r \neq \varnothing$. It is clear that any point in $\bigcap \tilde{H}_r$ is a fixed point of T. ▮

COROLLARY 6. *Let X, K, T, W, $LS(W, TW; K)$ be as in Theorem 2. If X is reflexive and satisfies Opial's condition (i.e., if z_n converges weakly to z and $z \neq v$ then* $\liminf_{n \to \infty} \|z_n - z\| < \liminf_{n \to \infty} \|z_n - v\|$*), then T has a fixed point.*

Proof. Immediate from Corollary 4 since if X satisfies Opial's condition and T is nonexpansive, then $I-T$ is demiclosed on K by a result of [12]. ▮

COROLLARY 7. *Let X, K, T, W, $LS(W, TW; K)$ be as in Theorem 2. If X is uniformly convex, then T has a fixed point.*

Proof. Immediate from Corollary 4 since if X is uniformly convex and T is nonexpansive, then $I-T$ is demiclosed on K by a result of [13]. ▮

3. LIPSCHITZIAN PSEUDO-CONTRACTIVE MAPPINGS

In this last section we give a result concerning lipschitzian pseudocontractive mappings, where we say that $T: K \to \mathscr{CB}(X)$ is pseudocontractive if for any x, $y \in K$, $u \in Tx$, $v \in Ty$, $r > 0$ we have

$$\|x-y\| \leqslant \|(1+r)(x-y) - r(u-v)\|.$$

This class of mappings includes, in the singlevalued case, all nonexpansive mappings, but for multivalued case this inclusion is not verified.

For example, the multivalued map $T: R \to \mathscr{K}(R)$ defined by $T(x) = [x, x+1]$ is nonexpansive but not pseudocontractive. (Here R denotes the real numbers.)

The results concerning the existence of fixed points for pseudocontractive mappings are closed related with the existence of zeros of accretive mappings (see Morales [14]) since T is pseudocontractive if and only if $I-T$ is accretive. The study of accretive mappings is very important in connection with the existence theory for nonlinear equations of evolution in Banach spaces (see, for example, 15–18]).

Theorem 1 (concerning nonexpansive multivalued self-mappings) is used in the proof of the following (concerning lipschitzian pseudocontractive multivalued mappings).

THEOREM 8. *Let X be a real Banach space whose bounded closed convex subsets have the fixed point property for multivalued nonexpansive point-compact self-mappings.*

Let K be a closed convex subset of X and let $T: K \to \mathscr{K}(X)$ be a lipschitzian pseudocontractive mapping which satisfies the inwardness condition $Tx \subseteq \overline{I_K(x)}$ for any $x \in K$.

Suppose that there exist $x_0 \in K$ and $\varepsilon > 0$ such that $LS((x_0, \overline{\mathrm{co}}(Tx_0)), \varepsilon; K)$ is bounded.

If $TK \cap B_r$ is compact for any $B_r := \{z \in K: \|z\| \leqslant r\}$ then T has a fixed point.

Proof. Let L be the Lipschitz constant of T and select α such that

$$0<\alpha<\min(L^{-1}, \varepsilon(2LH(\{x_0\}, Tx_0)+L\varepsilon)^{-1}).$$

Then for any $y \in K$ the mapping $T_y: K \to \mathscr{K}(X)$ given by $T_y(x) := (1-\alpha)\, y + \alpha Tx$ is a contraction that satisfies the inwardness condition $T_y(x) \subseteq \overline{I_K(x)}$ for any $x \in K$.

Hence, by [10, Theorem 3.4] the set $F_\alpha(y)$ of fixed points of T_y is non-empty, closed and it is easy to verify that

$$F_\alpha(y) \subseteq (1-\alpha)\, y + \alpha T F_\alpha(y) \tag{9}$$

$$y \in F_\alpha(y) \qquad \text{iff} \quad y \in Ty. \tag{10}$$

Moreover we will show that for $u, v \in K$ fixed

$$\|a-b\| \leqslant \|u-v\| \qquad \text{for any} \quad a \in F_\alpha(u),\ b \in F_\alpha(v). \tag{11}$$

Indeed $a \in F_\alpha(u)$ and $b \in F_\alpha(v)$ imply

$$a=(1-\alpha)u+\alpha\eta, \qquad \eta \in Ta \tag{12}$$

$$b=(1-\alpha)v+\alpha\xi, \qquad \xi \in Tb \tag{13}$$

and so, from pseudo-contractivity of T, from (12) and (13) and choosing $r<\alpha/(1-\alpha)$,

$$\begin{aligned}
\|a-b\| &\leqslant \|(1+r)(a-b)-r(\eta-\xi)\| \\
&= \left\|(1+r)(a-b)-\frac{r}{\alpha}\,(a-(1-\alpha)u-b+(1-\alpha)v)\right\| \\
&= \left\|(1+r)(a-b)-\frac{r}{\alpha}\,((a-b)-(1-\alpha)(u-v))\right\| \\
&= \left\|\left(1+r-\frac{r}{\alpha}\right)(a-b)-\frac{r}{\alpha}\,(1-\alpha)(u-v)\right\| \\
&\leqslant \left(1+r-\frac{r}{\alpha}\right)\|a-b\|+\frac{r}{\alpha}\,(1-\alpha)\,\|u-v\|,
\end{aligned}$$

i.e., $r(1-\alpha)\,\alpha^{-1}\,\|a-b\| \leqslant r(1-\alpha)\,\alpha^{-1}\,\|u-v\|$ proving (11).

It follows, in particular, that $F_\alpha(y)$ belongs to $\mathscr{CB}(K)$ for any $y \in K$. Besides $TF_\alpha(y)$ is bounded (since T is lipschitzian) and by (9) we can conclude, under the hypothesis $TK \cap B_r$ is compact for any r, that for any $y \in K$,

$$F_\alpha(y) \in \mathscr{K}(K). \tag{14}$$

Now we note that

$$H(F_\alpha(x), F_\alpha(y)) \leqslant \sup_{a \in F_\alpha(x),\, b \in F_\alpha(y)} \|a-b\|$$

and so from (11) and (14) it follows that the mapping

$$F_\alpha : K \to \mathscr{K}(K), \qquad x \to F_\alpha(x)$$

is nonexpansive and by (10) has the same fixed points of T.

Hence the claim will be proved if we are able to show that F_α has a fixed point.

From Theorem 1 it is enough to prove that the set $LS(x_0, \overline{\mathrm{co}}(F_\alpha(x_0)); K)$ is bounded. For this we will see that

$$LS(x_0, \overline{\mathrm{co}}(F_\alpha(x_0)); K) \subseteq LS(x_0, \overline{\mathrm{co}}(Tx_0), \varepsilon; K). \tag{15}$$

Indeed, from (9) it follows that

$$\overline{\mathrm{co}}(F_\alpha(x_0)) \subseteq (1-\alpha)\, x_0 + \alpha\, \overline{\mathrm{co}}(TF_\alpha(x_0))$$

and so $z \in LS(x_0, \overline{\mathrm{co}}(F_\alpha(x_0)); K)$ implies

$$\begin{aligned}
z &\in LS(x_0, (1-\alpha)\, x_0 + \alpha\, \overline{\mathrm{co}}(TF_\alpha(x_0)); K) \\
&= \bigcup_{y \in \overline{\mathrm{co}}(TF_\alpha(x_0))} LS(x_0, (1-\alpha)\, x_0 + \alpha y; K) \\
&= \bigcup_{y \in \overline{\mathrm{co}}(TF_\alpha(x_0))} \{z \in K : \tau(x_0 - z,\ (1-\alpha)\, x_0 + \alpha y - x_0) < 0\} \\
&= \bigcup_{y \in \overline{\mathrm{co}}(TF_\alpha(x_0))} \{z \in K : \tau(x_0 - z,\ \alpha(y - x_0)) < 0\} \\
&= \bigcup_{y \in \overline{\mathrm{co}}(TF_\alpha(x_0))} \{z \in K : \tau(x_0 - z,\ y - x_0) < 0\} \qquad \text{(from (3))} \\
&= \bigcup_{y \in \overline{\mathrm{co}}(TF_\alpha(x_0))} LS(x_0, y; K) \\
&= LS(x_0, \overline{\mathrm{co}}(TF_\alpha(x_0)); K).
\end{aligned}$$

Hence to prove (15) it is enough to prove that

$$LS(x_0, \overline{\mathrm{co}}(TF_\alpha(x_0)); K) \subseteq LS(x_0, \overline{\mathrm{co}}(Tx_0), \varepsilon; K). \tag{16}$$

First we note that

$$H(\overline{\mathrm{co}}(TF_\alpha(x_0)), \overline{\mathrm{co}}(Tx_0)) < \frac{\varepsilon}{2}. \tag{17}$$

Indeed

$$\begin{aligned}
&H(TF_\alpha(x_0), Tx_0)\\
&\quad \leqslant LH(F_\alpha(x_0), \{x_0\}) \qquad \text{(from [9, Theorem 2])}\\
&\quad = L \sup_{a \in F_\alpha(x_0)} \|x_0 - a\|\\
&\quad \leqslant L\alpha \sup_{a \in TF_\alpha(x_0)} \|a - x_0\| \qquad (\text{since } F_\alpha(x_0) - x_0 \subseteq \alpha(TF_\alpha(x_0)) - x_0)\\
&\quad = L\alpha H(TF_\alpha(x_0), \{x_0\})\\
&\quad \leqslant L\alpha(H(TF_\alpha(x_0), Tx_0) + H(Tx_0, \{x_0\}))
\end{aligned}$$

so that

$$H(TF_\alpha(x_0), Tx_0) \leqslant \frac{L\alpha}{1 - L\alpha} H(Tx_0, \{x_0\}) < \frac{\varepsilon}{2} \qquad (\text{by the choice of } \alpha).$$

From this (17) follows since

$$H(\overline{\text{co}}(TF_\alpha(x_0)), \overline{\text{co}}(Tx_0)) \leqslant H(TF_\alpha(x_0), Tx_0)$$

(see [9, Lemma 2]).

Finally we prove (16). Let $z \in LS(x_0, \overline{\text{co}}(TF_\alpha(x_0)); K)$. Then there exists $a \in \overline{\text{co}}(TF_\alpha(x_0))$ such that $\tau(x_0 - z, a - x_0) < 0$. From (17) it follows that there exists $b \in \overline{\text{co}}(Tx_0)$ such that $\|a - b\| < \varepsilon/2$ and so

$$\begin{aligned}
\tau(x_0 - z, b - x_0) &= \tau(x_0 - z, b - a + a - x_0)\\
&\leqslant \tau(x_0 - z, a - x_0) + \tau(x_0 - z, b - a) \qquad \text{(from (4))}\\
&< \tau(x_0 - z, b - a) \leqslant \|b - a\| \qquad \text{(from (2))}\\
&< \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

i.e., $z \in LS(x_0, \overline{\text{co}}(T(x_0)), \varepsilon; K)$. ∎

Remark 9. The previous theorem works also under the weaker hypothesis that T maps bounded sets into relatively compact sets instead of $TK \cap B_r$ compact for any $r > 0$.

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